

Asymptotic behavior and halting probability of Turing Machines*

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Abstract

Through a straightforward Bayesian approach we show that under some general conditions a maximum running time, namely the number of discrete steps performed by a computer program during its execution, can be defined such that the probability that such a program will halt *after* that time is smaller than any arbitrary fixed value. Consistency with known results and consequences are also discussed.

1 Introductory remarks

As it has been proved by Turing in 1936 [1], if we have a program p running on an Universal Turing Machine (UTM), then we have no general, finite and deterministic algorithm which allows us to know whether and when it will halt (this is the well known *halting problem*). That is to say that the halting behavior of a program, with the trivial exception of the simplest ones, is not computable and predictable by a unique, general procedure.

In this paper we show that, for what concerns the probability of its halt, every program running on an UTM is characterized by a peculiar asymptotic behavior in time. Similar results have been obtained by Calude *et al.* [2] and by Adamyan *et al.* [3] through a different approach, which makes use of quantum computation.

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2 Probabilistic approach

Given a program p of n bits, it is always possible to slightly change its code (increasing its size by a small, fixed amount of bits, let's say of s bits with $s \ll n$) in order to include a progressive, integer counter that starts counting soon after p starts to run on an UTM and stops, printing the total number of steps done by the UTM during the execution of p , soon after p halts. Let us call this new program p' . Its algorithmic size is then of $n + s$ bits.

As it has been said in the previous Section, for an arbitrary program of n bits no general, finite and deterministic procedure exists that allows us to know whether such program will ever halt or will keep running forever on an UTM [1]. Thus, in our case, we have no finite, deterministic algorithmic procedure to decide whether and when p' will print the integer number of steps done by the UTM during the execution of p , until the halt.

Let us now make a brief digression on algorithmic complexity. Suppose we have a (randomly chosen) binary string of n bits. Its algorithmic complexity (for a rigorous definition of algorithmic complexity, see [4]) is less than or equal to $n + c$ bits, where c is a specified constant in the chosen language. The case of a self-delimiting UTM (see [4]), for which the algorithmic complexity of a program of n bits is less than or equal to $n + O(\log_2 n)$ bits, is dealt with later. The *a priori* probability P_1 that its algorithmic complexity is equal to k bits¹, with $k \leq n + c$, is:

$$P_1 = \frac{2^k}{2^{n+c+1} - 2}, \quad (1)$$

since the total number of possible generating strings less than or equal to $n + c$ bits in size is $2^{n+c+1} - 2$, while there are only 2^k strings of k bits.

Now we want to calculate the somewhat different *a priori* probability P_2 that a (randomly chosen) program of algorithmic complexity of k bits that produces an output² generates just a string of n bits, with $n + c \geq k$.

From Bayes theorem we have that such probability can be calculated as the ratio between P_1 of eq. (1) and the sum of all the probability P_1 , with $n + c$ ranging from k to infinity:

$$P_2 = \frac{\frac{2^k}{2^{n+c+1}-2}}{\sum_{i=k}^{\infty} \frac{2^k}{2^{i+1}-2}}. \quad (2)$$

¹And thus, following the definition of algorithmic complexity [4], the *a priori* probability that a minimum program of k bits exists that generates the string of n bits as output.

²Note that this condition is crucial.

The probability P_2 can be easily rewritten and simplified as follows:

$$P_2 = \frac{1}{2^{n+c+1} - 2} \cdot \frac{1}{\sum_{i=k}^{\infty} \frac{1}{2^{i+1}-2}} \simeq \frac{2^k}{2^{n+c+1} - 2} \simeq \frac{1}{2^{n+c-k+1}} \quad n + c \geq k, \quad (3)$$

since the sum $\sum_{i=k}^{\infty} \frac{1}{2^{i+1}-2}$ is safely approximable to $1/2^k$ for $k \geq 10$.

Suppose now that we have a program p' of algorithmic complexity of k bits that will halt for sure, even if we are not able to know if and when it will do so by simply inspecting its code, and that contains a discrete step counter which counts the total number of steps done by the program until the halt and prints it as output³. Applying Bayes theorem again, the probability that the output string is of a size greater than or equal to m bits (with $m + c \geq k$) is:

$$P(\text{size} \geq m) = \frac{\sum_{s=m}^{\infty} \frac{2^k}{2^{s+c+1}-2}}{\sum_{i=k}^{\infty} \frac{2^k}{2^{i+1}-2}}, \quad m + c \geq k. \quad (4)$$

Using the same approximation done in eq. (3) the above equation becomes:

$$P(\text{size} \geq m) \simeq \frac{1}{2^{m+c-k}}, \quad m + c \geq k. \quad (5)$$

Thus, the probability that the output string is less than m bits in size is:

$$P(\text{size} < m) = 1 - \frac{\sum_{s=m}^{\infty} \frac{2^k}{2^{s+c+1}-2}}{\sum_{i=k}^{\infty} \frac{2^k}{2^{i+1}-2}} \simeq 1 - \frac{1}{2^{m+c-k}}, \quad m + c \geq k. \quad (6)$$

It is easy to verify through a numerical check that for $m + c \geq k + 50$, and thus even better for $m \geq k + 50$, the value of $P(\text{size} < m)$ is practically equal to 1.

This means that *if* a program p of algorithmic complexity of k bits halts, then it will do so in such a way that the total number of steps is a number of size less than or equal to $k + 50$ bits with very high probability. Thus, the decimal number of steps t done by the program before the halt (the discrete time before the halt, or the *characteristic time*) can not be more than 2^{k+51} , with very high probability.

The above analysis and results hold also if the program p has a size of k bits, but a smaller algorithmic complexity. In such cases the characteristic time 2^{k+51} provides a more conservative halting time.

³For every size k we can add few instructions to implement the counter, altering only slightly its length, as explained at the beginning of this Section.

If we consider a self-delimiting UTM, then the algorithmic complexity of a program of n bits is less than or equal to $n + O(\log_2 n)$ bits (see [4]). With such complexity measure eq. (1) becomes

$$P_1 = \frac{2^k}{2^{n+O(\log_2 n)+1} - 2}, \quad (7)$$

and eq. (2) consequently becomes

$$P_2 = \frac{\frac{2^k}{2^{n+O(\log_2 n)+1} - 2}}{\sum_{i=l}^{\infty} \frac{2^k}{2^{i+O(\log_2 i)+1} - 2}}, \quad (8)$$

where l is an integer such that $l + O(\log_2 l) = k$.

Equation (6) then becomes

$$P(\text{size} < m) = 1 - \frac{\sum_{s=m}^{\infty} \frac{2^k}{2^{s+O(\log_2 s)+1} - 2}}{\sum_{i=l}^{\infty} \frac{2^k}{2^{i+O(\log_2 i)+1} - 2}}, \quad m + O(\log_2 m) \geq k. \quad (9)$$

If we consider cases with $l \gg O(\log_2 l)$ and $m \gg O(\log_2 m)$, then eq.(9) can be safely approximable by

$$P(\text{size} < m) \simeq 1 - \frac{\sum_{s=m}^{\infty} \frac{2^k}{2^{s+1} - 2}}{\sum_{i=k}^{\infty} \frac{2^k}{2^{i+1} - 2}} \simeq 1 - \frac{1}{2^{m-k}}, \quad m \geq k. \quad (10)$$

Thus, for $l \gg O(\log_2 l)$, where $l + O(\log_2 l) = k$, and $m \gg O(\log_2 m)$, $P(\text{size} < m) \simeq 1 - 2^{k-m}$ and again the value of $P(\text{size} < m)$ is practically equal to 1 for $m \geq k + 50$, as before.

Actually, it is possible to prove⁴ that for a suitable constant $b > 1$ the function $1 - 2^{k-m+b}$ is a lower bound for the value of $P(\text{size} < m)$ for both complexity measures, and thus for $m \geq k + 50 + b$, the value of $P(\text{size} < m)$ becomes closer and closer to 1.

⁴Given eq. (6) or eq. (9), it is always possible to find a constant $b > 1$ such that:

$$1 - \frac{\sum_{s=m}^{\infty} \frac{2^k}{2^{s+1} - 2}}{\sum_{i=\frac{k}{b}}^{\infty} \frac{2^k}{2^{bi+1} - 2}} \simeq 1 - \frac{1}{2^{m-b-k}} \leq P(\text{size} < m) \leq 1.$$

For $m - b \geq k + 50$, and thus for $m \geq k + 50 + b$, the value of $P(\text{size} < m)$ is practically equal to 1.

3 Apparent paradoxes and their solution

Now, consider the following peculiar case, which obviously provides a limitation in the application of eq. (6), as every peculiar case does with probabilistic approaches.

It is possible to write a simple program p of $\lceil \log_2 n \rceil + c$ bits that, given the decimal number n , recursively calculates the number $2^{n+1} - 2$, counts from 1 to $2^{n+1} - 2$ and then stops. As it is obvious, the program p runs for a total number of steps greater than $2^{n+1} - 2$ and then stops, but eq. (6) may give an almost zero probability for such a result, since for suitable values of n the characteristic time $2^{\lceil \log_2 n \rceil + c + 51}$ can be much much smaller than $2^{n+1} - 2$.

As it always happens with probabilistic treatments, if, analyzing the code, we are able to know in advance that the program will halt and we are also able to know when it will halt, then eq. (6) is of poor use. But this does not dismiss our results as meaningless: it is obvious that when we have a deterministic solution of a problem, the probabilistic one simply does not apply.

Another apparent paradox with the above results is the following. It would seem possible to write a relatively small program h , again of $\lceil \log_2 n \rceil + c$ bits and with $n \gg \lceil \log_2 n \rceil + c$, which lists all the $2^{n+1} - 2$ programs of size less than or equal to n bits, runs each program for a discrete time equal to 2^{k+51} , where k is the size of the program, and stores the output strings of the halting ones. Then, simply printing as output a string greater by one unit than the greatest among those stored, the program h is able to provide a string of algorithmic complexity greater than or equal to n bits with a very high probability. This seems to challenge our results since $n \gg \lceil \log_2 n \rceil + c$ and h would be able to print a string of algorithmic complexity greater than its own size. But it does not.

As a matter of fact, among all the program of size less than or equal to n bits executed by h there is the program h itself, which we will call h_2 , and this fact generates a contradiction that does not allow the program h to produce any meaningful output.

In fact, the size of h_2 is obviously equal to $\lceil \log_2 n \rceil + c$ bits and we already know that after a characteristic time of $2^{\lceil \log_2 n \rceil + c + 51}$ steps it will be still running (the running time of h , and thus of h_2 , is obviously greater than 2^{n+51} steps). But h_2 should halt by definition and thus we *surely* have among the programs of size less than or equal to n selected by h as non halting, an halting one, making the procedure h meaningless.

Besides, note that if we accept the output s of h as true, then the output of h has to be at least $s + 1$, since also the output of h_2 is equal to s , and so on, endlessly.

Stated in other words, it is not possible to use our results to write a mechanical procedure able to print a string more complex than the mechanical procedure itself, as it should be according to the definition of algorithmic complexity.

4 Mathematical implications

Being able to solve the halting problem has unimaginable mathematical consequences since many unanswered mathematical problems, such as the Goldbach's conjecture, the extended Riemann hypothesis and others, can be solved if one is able to show whether the program written to find a single finite counterexample will ever halt [5, 6].

However, the estimate of the characteristic time done in the previous Section, namely $t \simeq 2^{n+51}$, where n is the size of the program, shows that the result obtained in eq. (6) is not much useful for the practical resolution of the halting problem, even for a probabilistic one, since almost all the interesting programs have a size much greater than 50 bits, giving astronomically huge characteristic times.

Anyway, our result should be of some theoretical interest since it shows an asymptotic behavior typical of every Turing Machine. All this seems to shed new light on the intrinsic significance of the algorithmic size or, more precisely, of the algorithmic complexity of a program encoding a mathematical problem. As a matter of fact, such a low-level and low-informative property of a program, as the number n of its bit-size, seems to be strongly related to its halting behavior, and thus, according to the above-mentioned mathematical connections, it seems to be intimately linked to the high level, mathematical truth encoded in the program. Calude *et al.* [7] have recently proposed a way to evaluate the difficulty of a finitely refutable mathematical problem which is based just on the algorithmic complexity, in a fixed language, of the Turing Machine encoding the problem.

Consider the Riemann hypothesis, for instance [8]. If I am able to show that a program of n bits, written to find a finite counterexample, will never halt with probability greater than 99.99999999%, then I may safely say that Riemann hypothesis is almost certainly true. The singular aspect here is that to be able to make such a claim I need only a finite number of numerical checks of the conjecture to reach a probability of 99.99999999%, out of an infinite number of zeros of the zeta function to be checked. Honestly speaking, it appears quite surprising.

The above argument could also be seen as a re-proposition of the Humian induction problem [9], this time applied to finitely refutable mathematical

statements: if a finitely refutable mathematical statement, encoded in a program of n bits of which we are not able to know if it will halt simply inspecting its code, holds true for about $t \simeq 2^{n+51}$ steps, then it is definitely true with a fixed arbitrary high probability. As a matter of fact, all this seems to give a strong quantitative support to the inspiring principles of “experimental mathematics”, proposed with force by many scholars in the last years [10].

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